functions P_i, where

$$\mu P_i = \frac{2}{\beta} \int\limits_0^{2\pi/\omega} \mu\Phi(\varphi_i^\circ, x_0, y_0)\,dt, \quad i = 1, 2, \ldots, 2N$$

whence we obtain

$$\mu P_i = -\frac{4\pi\omega v}{\beta R}\{D_{xn}\sin[\alpha_i + (-1)^n\lambda] - D_{yn}\cos[\alpha_i + (-1)^n\lambda]\} \tag{4}$$

The system of $2N$ equations $\mu P_i = 0$ for the important practical case of two balls in each selfbalancer ($N = 2$) gives 16 different solutions for the positions of the balls in the selfbalancers, of which the only solution corresponding to the absence of oscillations of the system refers to the case $D_{xn} = D_{yn} = 0$, or

$$\begin{aligned}\alpha_1 - \chi_1 &= -(\alpha_2 - \chi_1), \ \cos(\alpha_1 - \chi_1) = -D_1/(2mR)\\ \alpha_3 - \chi_2 &= -(\alpha_4 - \chi_2), \ \cos(\alpha_3 - \chi_2) = -D_2/(2mR)\end{aligned} \tag{5}$$

It can be seen from (5) that each selfbalancer compensates only the component of force F which rotates in the same direction as the corresponding SBS cage.

The common condition that the equilibrium positions of the balls be asymptotically stable is that the real parts of the roots of the fourth degree equation in z /2/

$$\det(\partial P_i/\partial\alpha_k - \delta_{ik}z) = 0;\ i, k = 1, \ldots, 4 \tag{6}$$

be negative. We can show by the Routh-Hurwitz method that the position of the balls given by (5) under the condition $D_n < 2mR$ is the only stable position in the range of rotation frequencies defined by the criterion

$$\operatorname{Im}\{\xi_c + j2\omega mN\} > 0 \tag{7}$$

which is the same as the criterion obtained in /1/ for the case of a single selfbalancer for extinguishing vibrations due to unbalance of a rotating rotor.

This conclusion was checked experimentally for the elementary case of a system consisting of a body on an isotropic elastic suspension (Fig.2) with natural frequency of oscillation ω_0; condition (7) corresponds to the inequality $\omega > \omega_0$.

REFERENCES

1. AGAFONOV YU.V., STudy of the stability of a ball selfbalancer of a rotor system on elastic supports, Mashinovedenie, 6, 1976.
2. BLEKHMAN I.I., Synchronization of dynamic systems, Nauka, Moscow, 1971.

Translated by D.E.B.

PMM U.S.S.R., Vol.51, No.3, pp.401-405, 1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

CONTROL OF THE SPECTRUM OF MULTIDIMENSIONAL OSCILLATORY OBJECTS*

V.A. BRUSIN

The solution is obtained in closed form of the problem of shifting in the complex plane any pairs by simple complex conjugate eigenvectors of the linear part of a controlled object by means of linear output variable feedbacks.

Consider the linear controlled object described by the equations

$$x^\cdot = Ax + Bu, \quad y = Cx \tag{1}$$

where x is the n-dimensional state vector, y is the m-dimensional output signal vector, u is the r-dimensional control vector, and A, B, C are $n\times n$, $n\times r$, and $m\times n$ matrices respectively.

A problem in stability and control theory concerns the control of the spectrum of a

*Prikl.Matem.Mekhan., 51,3,515-519,1987

closed system by means of output signal vector feedback. We know /1/ that, when $m = n, y = x$, when object (1) is completely controllable by means of linear feedback $u = Dx$, any desired distribution of the spectrum of the closed system can be realized. Several algorithms have been devized for finding the feedback matrix D, given the desired spectrum /2, 3/.

The problem is more complicated when the dimensionality m of the output signal vector is less than the dimensionality n of the state vector, as is basically the case in applied problems. No complete algorithmic solution of this problem has so far been obtained. Results have been obtained for various special cases. For instance, in /4/ the case $r = m$ and the class of proportionally integrating linear feedbacks

$$u = -\rho_1 Ky + \rho_2 \int_0^t Ly(\tau)\, d\tau$$

was considered (K, L are $m \times m$ and $\rho_{1,2}$ are positive numbers). It was shown that, in the "non-singular" case, the matrix K can be chosen in such a way that the m eigenvalues of the greatest real part of the stable matrix A shift leftwards for all sufficiently small $\rho_1 > 0$ when proprotional feedback is connected ($\rho_2 = 0$). (The results on the use of integrating feedback are concerned with the problem of neutralizing continuous interference). The solution obtained by perturbation theory for linear operators is also valid for an infinite-dimensional state space and has the form: $K = (C^+B^+)^{-1}$, where C^+, B^+ are the $(m \times m)$-matrices obtained by projection of the matrices C, B onto the m-dimensional eigensubspace of the matrix A corresponding to the eigenvalues considered. This solution exists if the matrices C^+, B^+ are non-singular.

Though fairly general, this result needs strengthening. First, it is not very convenient from the computational point of view. There are well-known difficulties in finding the eigenvectors of asymmetric matrices. Also, even small errors in finding them can lead to serious errors in finding the matrix $(C^+B^+)^{-1}$. Because of this, the solution given in /4/ cannot be regarded as stable to difficult kinds of errors, or to errors in finding the coefficients of the matrices A, B, C. Second, this solution does not describe the entire set of solutions having the required property. Third, the operator A is assumed in /4/ to be stable, which restricts the statement of the problem.

Below, we solve the problem on the "leftwards" shift of any q simple complex-conjugate eigenvalues of a matrix A of any type by means of the class of feedbacks of the type $u = -\rho Ky$, $\rho > 0$, and describe the set of matrices K which solve the problem for any sufficiently small $\rho > 0$. We use the method of D-splitting, /5/, so that only the characteristic polynomial of the matrix is taken as a basis. We basically consider the case of complex roots because this has a special role in problems of quenching the oscillations of multidimensional mechanical systems. (Consideration of the case of a zero eigenvalue has special features, due to the D-splitting /5/ have a singular line). Our approach can be extended to the case of distributed systems.

We assume that $\alpha_1 \pm i\omega_1, \alpha_2 \pm i\omega_2, \ldots, \alpha_q \pm i\omega_q\ (\omega_s > 0;\ s = 1, \ldots, q)$ are simple eigenvalues of the matrix A. We consider the class of feedbacks (controllers)

$$u = -K(\rho)\, y$$

$$K = \left\| \begin{matrix} k_{11} & \ldots & k_{1m} \\ \ldots & \ldots & \ldots \\ k_{r1} & \ldots & k_{rm} \end{matrix} \right\|, \quad k_{ji} = k_{ij}(\rho), \quad k_{ij}(0) = 0 \tag{2}$$

($k_{ij}(\rho)$ are functions of the parameter ρ, differentiable at zero). We denote by

$$D(p, K) = \det \left\| \begin{matrix} -pE_n + A & B \\ KC & E_r \end{matrix} \right\| = p^n + d_1(K)\, p^{n-1} + \ldots + d_n(K)$$

the characteristic polynomial of the linear system (1), (2).

We introduce the polynomial

$$D_s(p, K) = D(p + \alpha_s, K) \tag{3}$$

(Here and throughout, $s = 1, \ldots, q$). Obviously, it has a pair of pure imaginary roots $\pm i\omega_s$. We form the complex function of a real variable

$$D_s(i\omega, K) = P_s(\omega^2, K) + i\omega Q_s(\omega^2, K) \tag{4}$$

where P_s and Q_s are polynomials in ω^2 with real coefficients, dependent on the coefficients $k_{ij}(\rho)$ of the feedback. We Taylor-expand the function $P_s(\omega^2, K), Q_s(\omega^2, K)$ at the point $\omega^2 = \omega_s^2$, $k_{ij} = 0$ and denote by $\Delta P_s, \Delta Q_s$ the linear parts of this expansion:

$$\Delta P_s = \sum_{i=1}^{r} \sum_{j=1}^{m} p_{ijs} k_{ij} + p_{0s} \Delta\omega^2$$

$$\Delta Q_s = \sum_{i=1}^{r} \sum_{j=1}^{m} q_{ijs} k_{ij} + q_{0s} \Delta\omega^2, \quad \Delta\omega^2 = \omega^2 - \omega_s^2, \tag{5}$$

$$p_{ijs} = \frac{\partial P_s}{\partial k_{ij}}, \quad q_{ijs} = \frac{\partial Q_s}{\partial k_{ij}}, \quad p_{0s} = \frac{\partial P_s}{\partial (\omega^2)}, \quad q_{0s} = \frac{\partial Q_s}{\partial (\omega^2)} \tag{6}$$

where the values of the derivatives are taken for $k_{ij} = 0, \Delta\omega^2 = 0$.

We order the elements of the matrices $k_{11}, k_{12}, \ldots, k_{rm}$, first selecting elements of the first row, then of the second, etc. This ordered set of elements $k_{ij}(\rho)$ forms the mr-dimensional vector $\mathbf{k}(\rho)$. By (6), we form the q pieces of mr-dimensional vectors

$$\mathbf{N}_s = \left(\begin{vmatrix} p_{11s} & p_{0s} \\ q_{11s} & q_{0s} \end{vmatrix}, \quad \begin{vmatrix} p_{12s} & p_{0s} \\ q_{12s} & q_{0s} \end{vmatrix}, \ldots, \begin{vmatrix} p_{rms} & p_{0s} \\ q_{rms} & q_{0s} \end{vmatrix} \right)$$

(the elements p_{ij}, q_{ij} are written in the same order as the components k_{ij} of the vector $\mathbf{k}$).

Theorem. Let

$$\operatorname{rank} \begin{Vmatrix} p_{11s} p_{12s} & \cdots & p_{rms} \\ q_{11s} q_{12s} & \cdots & q_{rms} \end{Vmatrix} = 2 \tag{7}$$

$$\langle \mathbf{N}_s, \mathbf{k} \rangle > 0 \tag{8}$$

Then, if $\mathbf{k}'(0) = \mathbf{k}$, a number $\bar{\rho} > 0$ exists such that, for all $\rho \in (0, \bar{\rho})$, the feedback (2) provides a leftwards shift of the eigenvalues $\alpha_s \pm i\omega_s$ in the complex plane.

If one of the scalar products (8) has a negative sign, then, for sufficiently small $\rho > 0$, the corresponding eigenvalue will shift rightwards.

Notes. 1°. If the domain of parameters k_{ij}, given by inequalities (8), is not empty, we can obviously choose the parameters in such a way that small errors in finding the parameters of the controlled system do not violate (8). If upper and lower bounds of the errors are known, we can use (8) to make a choice of k_{ij} for which the errors cannot affect the leftwards shift of the eigenvalues.

2°. Assume that we aim to shift the simple complex conjugate eigenvalues of the matrix A in such a way that their real parts are less than a given number $-\delta < 0$. We first choose the eigenvalues which have the greatest real part (there may be one or more of such complex conjugate pairs). For the set of eigenvalues thus chosen, we use the theorem to find the vector $\mathbf{k}_0$, then the matrix $K_0(\rho)$ of feedbacks. Then, by increasing ρ, we move these eigenvalues leftwards until the maximum value of the real parts of all the complex conjugate roots is reduced. Let ρ_0 be the limiting value of ρ. If this ρ still does not achieve our aim, we proceed as follows. We put $u = -K_0(\rho_0) y + u_1$ and consider a new linear object with the same output signal vector and a new control vector u_1. For the new object we repeat our procedure, defining $u_1 = -K_1(\rho_1) y + u_2$.

This procedure recalls the familiar method of steepest descent when minimizing a function of several variables numerically. In the latter method, we first find the vector gradient at each step, then make a displacement in the corresponding direction until the "descent" occurs.

3°. It can happen, however, that N-tuple repetition of the procedure leads to a linear object with control u_N, for which further use of the procedure is impossible, though its aim is not achieved. We then have to try to achieve our aim by extending the class of feedbacks, by introducing say feedbacks of the type

$$u = -K(\rho) y + L(\rho) v, \quad v^{\cdot} = Rv + Sy, \quad R = \operatorname{diag}(r_1, \ldots, r_m) \tag{9}$$

where $r_i < 0$ is a fairly small number.

The introduction of feedbacks (controllers) of type (9) is equivalent to increasing the dimensionality of the initial control object.

For, if we introduce new variables into the state vector and denote by $X = \operatorname{col}(x, v)$ the new state vector, the equation of the extended object can be written as

$$X^{\cdot} = \begin{Vmatrix} A & 0 \\ SC & R \end{Vmatrix} X + \begin{Vmatrix} B \\ 0 \end{Vmatrix} u$$

If $\dot{Y} = \operatorname{col}(Cx, v) = FX, F = (C, E_m)$ (E_m is the $m \times m$) identity matrix) denotes the extended object output vector, we arrive at the initial statement of the problem for it, i.e., we can use our theorem.

4°. The theorem can be extended to infinite-dimensional dynamic systems with a characteristic polynomial which is a smooth function of p and of the feedback parameters in the neighbourhood of the roots.

Proof of the theorem. We introduce the $m \times r$ dimensional space with orthonormalized basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{mr}$. We shall regard k_{ij} as the coordinates of a vector $\mathbf{k}$ in this space, i.e., k_{ij} is the projection of $\mathbf{k}$ onto the vector $\mathbf{e}_{N(i,j)}$, $N(i, j) = (i-1)m + j$, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant m$, $1 \leqslant$

$N \leqslant mr$.

The n-th degree polynomial $D_s(p, K)$, whose coefficients depend on k_{ij}, defines a division of the space into domains $D_s[l, n-l]$ $(l = 0, 1, \ldots, n)$, in each of which the polynomial has l roots with negative real part and $n-l$ roots with positive real part. This division is called a D-division of the parameter space of the polynomial /5/. The boundary of the D-division is a hypersurface G_s in the space, defined by the equation in parametric form

$$P_s(\omega^2, K) = 0, \quad \omega Q_s(\omega^2, K) = 0, \quad -\infty < \omega < +\infty \tag{10}$$

where P_s, Q_s are given in (4).

When $\omega^2 = \omega_s^2$ the surface passes through the origin. We take a sufficiently small neighbourhood of the surface at the origin, with ω^2 sufficiently close to ω_s^2. We denote by double shading /5/ the side of the piece of surface G_s,adjacent to the domain $D_s[l, n-l]$ with the larger value of l. Then, since the root $\alpha_s \pm i\omega_s$ is simple, the other unshaded side on this piece of surface is adjacent to the domain $D_s[l-2, n-l+2]$.

We introduce the tangent hyperplane at the origin to the piece of surface G_s. By (5) and (6), the equation of this hyperplane is

$$\sum_{i=1}^{r}\sum_{j=1}^{m} p_{ijs}k_{ij} + p_{0s}\Delta\omega^2 = 0, \quad \sum_{i=1}^{r}\sum_{j=1}^{m} q_{ijs}k_{ij} + q_{0s}\Delta\omega^2 = 0 \tag{11}$$

We put

$$\Delta_{s, N(i,j)} = \begin{vmatrix} p_{ijs} & p_{0s} \\ q_{ijs} & q_{0s} \end{vmatrix}, \quad N(i, j) = 1, \ldots, mr$$

Then, if $\Delta_{s,1}^2 + \Delta_{s,2}^2 + \ldots + \Delta_{s,mr}^2 \neq 0$, Eq.(11) defines a hyperplane of co-dimension unity, the normal to it being the vector

$$\mathbf{N}_s = \pm(\Delta_{s,1}, \Delta_{s,2}, \ldots, \Delta_{s,mr}) \tag{12}$$

We choose the sign in (12) so that vector $\mathbf{N}_s$ is directed towards the side with double shading, i.e., into the domain $D_s[l, n-l]$ with the larger value of l.

To be specific, let

$$\Delta_{12s} = \begin{vmatrix} p_{11s} & p_{12s} \\ q_{11s} & q_{12s} \end{vmatrix} \neq 0$$

We then consider the section S_{12} of the piece of hypersurface G_s by the hyperplane $k_{11} = k_1, k_{12} = k_2, k_{ij} = 0, (i, j) \neq (1, 1), (1, 2)$ of dimensionality two. The tangent line at the origin (the trace of the tangent hyperplane (11)) is described in the plane of the section by the equations (the subscript s is now omitted)

$$p_{11}k_1 + p_{12}k_2 + p_0\Delta\omega^2 = 0, \quad q_{11}k_1 + q_{12}k_2 + q_0\Delta\omega^2 = 0 \tag{13}$$

From (13) we have

$$k_1 = \frac{\Delta_2}{\Delta_{12}}\Delta\omega^2, \quad k_2 = -\frac{\Delta_1}{\Delta_{12}}\Delta\omega^2 \tag{14}$$

Denote by γ_{12} the vector $(\Delta_2/\Delta_{12}, -\Delta_1/\Delta_{12}, 0, \ldots, 0)$. By (13) and (14), it is the tangent vector (at the origin) to the section S_{12}, the direction of this vector being the same as the direction of increase of the parameter ω^2 in this section. We also carrier the vector $\mathbf{N}_{12} = \pm(\Delta_1, \Delta_2, 0, \ldots, 0)$. Obviously, $\mathbf{N}_{12}$ is the projection of the vector N onto the plane of section S_{12}, while $\langle \mathbf{N}_{12}, \gamma_{12}\rangle = 0$.

Let $\Delta_{12} > 0$. Then, on the piece of curve considered, the S_{12} shading is on the left, if we move along this curve in the direction of increasing ω^2 /5/. Hence, in the plane of the section, the shortest rotation from the vector γ_{12} to the vector $\mathbf{N}_{12}$ must be counter clockwise, since it is then that the vector $\mathbf{N}_{12}$ (and hence N also) is directed towards the shading. Using vector multiplication and the expressions for $\gamma_{12}, \mathbf{N}_{12}$, we can see that the vector $\mathbf{N}_{12}$ will satisfy this condition if we choose the plus sign. Hence the required direction of the vector N is obtained if we take the plus sign in (12). If $\Delta_{12} < 0$, similar arguments lead us to the same result.

In short, if the vector of parameters satisfies the inequality $\langle \mathbf{N}_s, \mathbf{k}\rangle > 0$ (with the indicated choice of sign in the expression for $\mathbf{N}_s$), the vector **k** will be directed towards the shading. An conversely, if $\langle \mathbf{N}_s, \mathbf{k}\rangle < 0$, it will be in the opposite direction. By the conditions of the theorem, in the first case, as ρ increases at the point $\rho = 0$, the eigenvalues $\alpha_s \pm i\omega_s$ will shift "leftwards" in the complex plane, or in the second case, "rightwards." The theorem is proved.

REFERENCES

1. KALMAN R.E. et al., Topics in mathematical system theory, McGraw-Hill, 1968.

2. ANDREYEV YU.N., Control of finite-dimensional linear objects, Nauka, Moscow, 1976.
3. GELIG A.KH., LEONOV G.A. and YAKUBOVICH V.A., Stability of non-linear systems with a non-unique equilibrium state, Nauka, Moscow, 1978.
4. POHJOLAINEN S.A., Robust multivariable PI-controller for infinite dimensional systems, Trans. IEEE, Ser. AC, 27, 1, 1982.
5. NEIMARK YU.I., Dynamic systems and controlled processes (Dinamicheskie sistemy i upravlyaemye protsesssy), Nauka, Moscow, 1978.

Translated by D.E.B.

PMM U.S.S.R.,Vol.51,No.3,pp.405-407,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

ANALYTICAL SOLUTION OF A FLOW PROBLEM IN THE NEIGHBOURHOOD OF THE BOUNDARY LAYER SEPARATION POINT ON A MOVING SURFACE*

V.V. SYCHEV

An accurate solution to a previously formulated boundary value problem /1/ for the boundary layer (BL) equations describing flow in the neighbourhood of the separation point on a moving surface is obtained.

The plane stationary flow of a viscous incompressible liquid in the neighbourhood of a release point on a surface which is moving downstream at a constant velocity is examined. As a result of the asymptotic analysis of the Navier-Stokes equations with large Reynolds numbers (R) it has been established /2/ that in the neighbourhood of the release point there is a region of interaction between the BL and the outer potential flow where a large unfavourable selfinduced pressure gradient is acting (the longitudinal and transverse dimensions of this region are quantities of the order of $R^{-1/3}$, see Fig.1). Upstream of this region, the flow is described by the BL equations; the pressure distribution outside this region is given (locally) by the solution of the theory of potential flows of an ideal liquid with free streamlines. The selfinduced pressure gradient leads to intense deceleration of the liquid inside the BL but does not cause flow separation, i.e. the appearance of a return flow in the interaction region /3/.

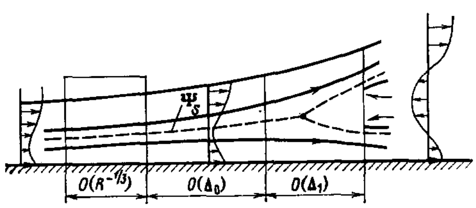

Fig.1

Subsequent analysis showed /1/ that the separation point must lie in a region situated inside the BL, at a short distance upstream of the interaction region. The asymptotic presentation of the solution of the Navier-Stokes equations (as $R \to \infty$) for this region take the form

$$\begin{aligned}
&x = L\Delta_1 x', \quad y = LR^{-1/2}y' \\
&\psi = U_{00}LR^{-1/2}\,[\Psi_s + \Delta_1\psi_0\,(x', y') + \ldots] \\
&p = p_{00} + \rho U_{00}^2\,[\Delta_1^2 p_0\,(x') + \ldots] \\
&\Delta_1 = \sigma^{1/2}R^{-1/12}, \ \sigma\,(\ln\sigma^{-1})^{2/3} = R^{-1/3}, \ R = LU_{00}/\nu
\end{aligned} \tag{1}$$

*Prikl.Matem.Mekhan.,51,3,519-521,1987